Lecture 1

Andrei Antonenko

January 30, 2003

1 Introduction

The first mathematical object which every person meets even in the childhood is a *number*. So, the number is often considered as a main mathematical object, which is not really true (though, it is not too far from reality). In this lecture we'll construct main groups (*sets*) of numbers which we will use during the course, and figure out, what is the main algebraic object of algebraic studies.

2 Different types of numbers

Numbers appeared during the time of ancient civilizations — but there numbers were simply used for counting, so, only simplest numbers were used, like 1, 2, 3, 4, etc. These numbers are called *natural*. We can perform simple operations with these numbers — we can add them together, and we can multiply them — we'll still get natural number. But we can not subtract them! How can we subtract, for example, 2 from 1? The answer will not be natural, so we need to extend natural numbers by 0 and negative numbers, like -1, -2, -3, etc. So, we'll get *integer* numbers. Now we can subtract, but there is another problem — we can not still divide numbers! For example, we can not divide 1 by 2 — we will not get an integer number! So, we have to introduce another class of numbers — *rational* numbers. Rational numbers are the numbers which can be represented as $\frac{m}{n}$, where m is integer and n is natural, and we can always find such numbers m and n that they do not have any common divisor greater than 1.

Example 2.1. $\frac{1}{2} = \frac{2}{4} = \frac{100}{200}$. *Numbers 1 and 2 do not have any common divisors greater than 1, numbers 2 and 4 have common divisor which is equal to 2, and numbers 100 and 200 have common divisor which is equal to 100.*

Now we have 4 elementary operations: "+", "-", "*", "/". Let's define the operation of taking a *power*. By definition $a^b = \underbrace{a \cdot a \cdot \dots \cdot a}_{b}$ for any a and natural b. So, we can take a second power of a number (*square*), but can we always find a number with the given square? For example, can we find a number x such that $x \cdot x = 2$?

Theorem 2.2. *There is no rational number such that its square equals to 2.*

Proof. Let's prove this by contradiction. Let α be such number that $\alpha^2 = 2$, and m and n are minimal integer numbers such that $\alpha = \frac{m}{n}$ (they are minimal, so they do not have any common divisor which is greater than 1) . So,

$$\alpha^2 = \frac{m^2}{n^2} = 2.$$

So, $m^2 = 2n^2$. First of all, m should be even (Otherwise, if m is odd, then m^2 is odd, which is not true). So, $m = 2k$, and $m^2 = 4k^2$, $4k^2 = 2n^2$, and $2k^2 = n^2$. From this it follows that n is even too, thus it can be represented as $n = 2l$, and thus we see that $\alpha = \frac{m}{n} = \frac{2k}{2l} = \frac{k}{l}$. So, we found another representation of α, and we have that $k < m$ and $l < n$, which contradicts the initial assumption of the minimality of m and n. This contradiction proves that $\sqrt{2}$ can not be represented as a fraction, and thus is not a rational number. □

So, we found a number which is not rational. Such numbers which can not be written as $\frac{m}{n}$ for integer m and natural n are called *irrational*.

If we take these two groups of numbers — rational and irrational, and combine them together, we will get the class of all numbers we can think of — class of *real* numbers.

Actually, we can proceed with constructing different type of numbers, since, for example, we still can not find a number x, such that $x^2 = -1$. If we go by this way, we'll come up with the notion of complex number. But we don't need it right now.

3 Sets and operations

Bunches of similar objects are called *sets*. The exact mathematical definition of the set is very-very complicated, and is not really needed. But we will use the word "set" without this exact definition.

Sets can contain objects of various types. We can introduce a set of all the students of Stony Brook University, or the set of all skyscrapers in the New York City. These sets are *finite*, since they contain only finite number of elements, but some sets are *infinite*, like a set of natural numbers.

Different sets of numbers which we introduces above are denoted in the following way:

- $\mathbb{N}$ — set of natural numbers
- $\mathbb{Z}$ — set of integer numbers

- $\mathbb{Q}$ — set of rational numbers
- $\mathbb{R}$ — set of real numbers
- $\mathbb{C}$ — set of complex numbers

But in mathematics it's not interesting to study just sets. We study sets with some operations on them, for example, set $\mathbb{N}$ with operation of addition or $\mathbb{Q}$ with operation of multiplication.

In general, *operation* is a rule how to get an element of the set corresponding to 2 given elements of the set. For example, in case of addition in the set $\mathbb{R}$ we should add 2 given numbers together to get the result.

The main object of algebraic studies is the set with given operation(s) (maybe, several operations). Set with operation on it is called algebraic structure.

Now, I'll tell few more words about the notation of sets. In some cases we will need to enumerate all the elements of the set. Then we will write:

$$A = \{1, 2, 3, 4, 10\}.$$

It will mean that the set A contains 5 elements — 1, 2, 3, 4, 10. Following notation is used very often: if an element a belongs to the set A then we'll write $a \in A$. For example, $2 \in A$. If an element a doesn't belong to the set A then we'll write $a \notin A$. For example, $6 \notin A$.

The set contains only different elements, it means that it is not good to write that $A = \{1, 1, 1, 1, 2\}$. This set will be the same as $B = \{1, 2\}$, and both of them will contain 2 elements. Moreover, the set is unordered: set $A = \{1, 2\}$ equals to the set $B = \{2, 1\}$.

The number of elements of the set A is called *cardinality* of the set and is denoted by $|A|$. Cardinality can be either finite or infinite, for example, $|\{1, 3, 5, 7\}| = 4$, and $|\mathbb{N}| = \infty$.

The main operation on sets are *intersection* and *union*.

Definition 3.1. The union *of 2 sets A and B $A \cup B$ is the set consisting of elements belonging to either A or B.*

Definition 3.2. The intersection *of 2 sets A and B $A \cap B$ is the set consisting of elements belonging to both A and B.*

Example 3.3. *Let $A = \{1, 2, 3\}$, and $B = \{2, 3, 4, 5\}$. Then $A \cup B = \{1, 2, 3, 4, 5\}$ and $A \cap = \{2, 3\}$.*